MATHEMATICAL APPENDIX

to

Principles of Economics

by

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(http://www.econlib.org/library/Marshall/marPtoc.html)

NOTE I. (p. 93). The law of diminution of marginal utility may be expressed thus:—If u be the total utility of an amount x of a commodity to a given person at a given time, then marginal utility is measured by $\frac{du}{dx} \cdot \delta x$; while $\frac{du}{dx}$ measures the *marginal degree* of utility. Jevons and some other writers use "Final utility" to indicate what Jevons elsewhere calls Final degree of utility. There is room for doubt as to which mode of expression is the more convenient: no question of principle is involved in the decision. Subject to the qualifications mentioned in the text $\frac{d^2u}{dx^2}$ is always negative.

NOTE II. (p. 96). If m is the amount of money or general purchasing power at a person's disposal at any time, and μ represents its total utility to him, then $\frac{d\mu}{dm}$ represents the marginal degree of utility of money to him.

If p is the price which he is just willing to pay for an amount x of the commodity which gives him a total pleasure u, then

$$\frac{d\mu}{dm}\Delta p = \Delta u; \text{ and} \frac{d\mu}{dm}\frac{dp}{dx} = \frac{du}{dx}.$$

If p' is the price which he is just willing to pay for an amount x' of another commodity, which affords him a total pleasure u', then

$$\frac{d\mu}{dm} \cdot \frac{dp'}{dx'} = \frac{du'}{dx'};$$

and therefore

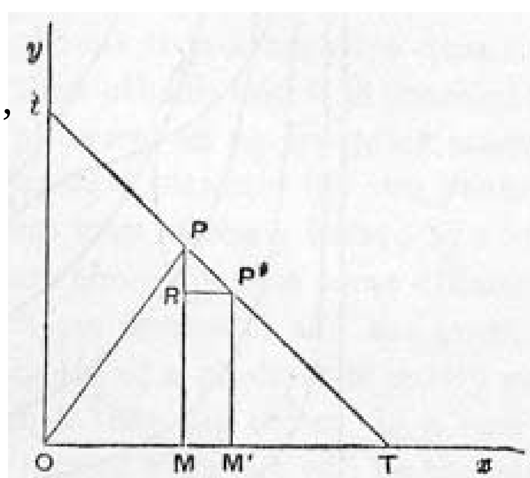
$$\frac{dp}{dx} : \frac{dp'}{dx'} = \frac{du}{dx} : \frac{du'}{dx'}.$$

(Compare Jevons' chapter on the *Theory of Exchange*, p. 151.)

Every increase in his means diminishes the marginal degree of utility of money to him; that is, $\frac{d^2\mu}{dm^2}$ is always negative.

Therefore, the marginal utility to him of an amount x of a commodity remaining unchanged, an increase in his means increases $\frac{du}{dx} \div \frac{d\mu}{dm}$; i.e. it increases $\frac{dp}{dx}$, that is, the rate at which he is willing to pay for further supplies of it. We may regard $\frac{dp}{dx}$ as a function of m, u, and x; and then we have $\frac{d^2p}{dm\,dx}$ always positive. Of course $\frac{d^2p}{du\,dx}$ is always positive.

Note III. (p. 103). Let P, P' be consecutive points on the demand curve; let PRM be drawn perpendicular to Ox, and let PP' cut Ox and Oy in T and t respectively; so that $P'R$ is that increment in the amount demanded which corresponds



to a diminution PR in the price per unit of the commodity.

Then the elasticity of demand at P is measured by

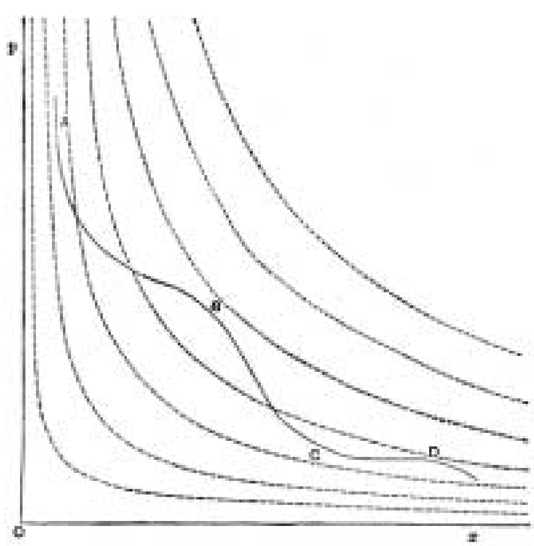
$$\frac{P'R}{OM} \div \frac{PR}{PM}, \qquad \text{i.e. by } \frac{P'R}{PR} \times \frac{PM}{OM};$$
$$\text{i.e. by } \frac{TM}{PM} \times \frac{PM}{OM},$$
$$\text{i.e. by } \frac{TM}{OM} \text{ or by } \frac{PT}{Pt}.$$

When the distance between P and P' is diminished indefinitely, PP' becomes the tangent; and thus the proposition stated on p. 103 is proved.

It is obvious *à priori* that the measure of elasticity cannot be altered by altering relatively to one another the scales on which distances parallel to Ox and Oy are measured. But a geometrical proof of this result can be got easily by the method of projections: while analytically it is clear that $\frac{dx}{x} \div \frac{-dy}{y}$, which is the analytical expression for the measure of elasticity, does not change its value if the curve $y = f(x)$ be drawn to new scales, so that its equation becomes $qy = f(px)$; where p and q are constants.

If the elasticity of demand be equal to unity for all prices of the commodity, any fall in price will cause a proportionate increase in the amount bought, and therefore will make no change in the total outlay which purchasers make for the commodity. Such a demand may therefore be called a *constant outlay demand*. The curve which represents it, a *constant outlay curve*, as it may be called, is a rectangular hyperbola with Ox and Oy as asymptotes; and a series of such curves are represented by the dotted curves in the following figure.

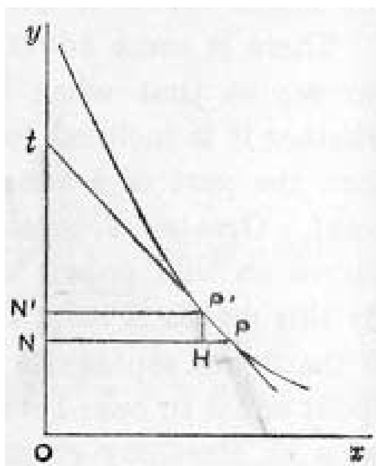
There is some advantage in accustoming the eye to the shape of these curves; so that when looking at a demand curve one can tell at once whether it is inclined to the vertical at any point at a greater or less angle than the part of a constant outlay curve, which would pass through that point. Greater accuracy may be obtained by tracing constant outlay curves on thin paper, and then laying the paper over the demand curve. By this means it may, for instance, be seen at once that the demand curve in the figure represents at each of the points A, B, C and D an elasticity about equal to one: between A and B, and again between C and D, it represents an elasticity greater than one; while between B and C it represents an elasticity less than one. It will be found that practice of this kind makes it easy to detect the nature of the assumptions with regard to the character of the demand for a commodity, which are implicitly made in drawing a demand curve of any particular shape; and is a safeguard against the unconscious introduction of improbable assumptions.



The general equation to demand curves representing at every point an elasticity equal to n is $\frac{dx}{x} + n\frac{dy}{y} = 0$, i.e. $xy^n = C$.

It is worth noting that in such a curve $\frac{dx}{dy} = -\frac{C}{y^{n+1}}$; that is, the proportion in which the amount demanded increases in consequence of a small fall in the price varies inversely as the $(n+1)^{\text{th}}$ power of the price. In the case of the constant outlay curves it varies inversely as the square of the price; or, which is the same thing in this case, directly as the square of the amount.

NOTE IV. (p. 110). The lapse of time being measured downwards along Oy; and the amounts, of which record is being made, being measured by distances from Oy; then P' and P being adjacent points on the curve which traces the growth of the amount, the rate of increase in a small unit of time $N'N$ is



$$\frac{PH}{P'N'} = \frac{PH}{P'H} \cdot \frac{P'H}{P'N'} = \frac{PN}{Nt} \cdot \frac{P'H}{P'N'} = \frac{P'H}{Nt};$$

since PN and $P'N'$ are equal in the limit.

If we take a year as the unit of time we find the annual rate of increase represented by the inverse of the number of years in Nt.

If Nt were equal to c, a constant for all points of the curve, then the rate of increase would be constant and equal to $\frac{1}{c}$. In this case $-x\dfrac{dy}{dx} = c$ for all values of x; that is, the equation to the curve is $y = a - c\log x$

NOTE V. (p. 123). We have seen in the text that the rate at which future pleasures are discounted varies greatly from one individual to another. Let r be the rate of interest per annum, which must be added to a present pleasure in order to make it just balance a future pleasure, that will be of equal amount to its recipient, when it comes; then r may be 50 or even 200 per cent. to one person, while for his neighbour it is a negative quantity. Moreover some pleasures are more urgent than others; and it is conceivable even that a person may discount future pleasures in an irregular random way; he may be almost as willing to postpone a pleasure for two years as for one; or, on the other hand, he may object very strongly indeed to a long postponement, but scarcely at all to a short one. There is some difference of opinion as to whether such irregularities are frequent; and the question cannot easily be decided; for since the estimate of a pleasure is purely subjective, it would be difficult to detect them if they did occur. In a case in which there are no such irregularities, the rate of discount will be the same for each element of time; or, to state the same thing in other words, it will obey the exponential law. And if h be the future amount of a pleasure of which the probability is p, and which will occur, if at all, at time t; and if $R = 1+r$; then the present value of the pleasure is phR^{-t}. It must, however, be borne in mind that this result belongs to Hedonics, and not properly to Economics.

Arguing still on the same hypothesis we may say that, if ϖ be the probability that a person will derive an element of happiness, Δh, from the possession of, say, a piano in the element of time

Δt, then the present value of the piano to him is $\int_0^T \varpi R^{-t}\frac{dh}{dt}dt$. If we are to include all the happiness that results from the event at whatever distance of time we must take $T = \infty$. If the source of pleasure is in Bentham's phrase "impure," $\frac{dh}{dt}$ will probably be negative for some values of t; and of course the whole value of the integral may be negative.

NOTE VI. (pp. 132, 3). If y be the price at which an amount x of a commodity can find purchasers in a given market, and $y = f(x)$ be the equation to the demand curve, then the total utility of the commodity is measured by $\int_0^a f(x)dx$, where a is the amount consumed.

If however an amount b of the commodity is necessary for existence, $f(z)$ will be infinite, or at least indefinitely great, for values of x less than b. We must therefore take life for granted, and estimate separately the total utility of that part of the supply of the commodity which is in excess of absolute necessaries: it is of course $\int_b^a f(x)dx$.

If there are several commodities which will satisfy the same imperative want, as e.g. water and milk, either of which will quench thirst, we shall find that, under the ordinary conditions of life, no great error is introduced by adopting the simple plan of assuming that the necessary supply comes exclusively from that one which is cheapest.

It should be noted that, in the discussion of consumers' surplus, we assume that the marginal utility of money to the individual purchaser is the same throughout. Strictly speaking we ought to take account of the fact that if he spent less on tea, the marginal utility of money to him would be less than it is, and he would get an element of consumer's surplus from buying other things at prices which now yield him no such rent. But these changes of consumers' rent (being of the second order of smallness) may be neglected, on the assumption, which underlies our whole reasoning, that his expenditure on any one thing, as, for instance, tea, is only a small part of his whole expenditure. (Compare Book V. ch. II. §3.) If, for any reason, it be desirable to take account of the influence which his expenditure on tea exerts on the value of money to him, it is only necessary to multiply $f(x)$ within the integral given above by that function of $xf(x)$ (i.e. of the amount which he has already spent on tea) which represents the marginal utility to him of money when his stock of it has been diminished by that amount.

NOTE VII. (p. 134). Thus if $a_1, a_2, a_3, \ldots$ be the amounts consumed of the several commodities of which $b_1, b_2, b_3, \ldots$ are necessary for existence, if $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x), \ldots$ be the equations to their demand curves and if we may neglect all inequalities in the distribution of wealth; then the total utility of income, subsistence being taken for granted, might be represented by $\Sigma \int_b^a f(x)dx$, if we could find a plan for grouping together in one common demand curve all those things which satisfy the same wants, and are rivals; and also for every group of things of which the services are complementary (see Book V. ch. VI). But we cannot do this: and therefore the formula remains a mere general expression, having no practical application. See footnote on pp. 131, 2; also the latter part of Note XIV.

NOTE VIII. (p. 135). If y be the happiness which a person derives from an income x; and if, after Bernoulli, we assume that the increased happiness which he derives from the addition of one per cent. to his income is the same whatever his income be, we have $x\dfrac{dy}{dx} = K$, and $\therefore$ $y = K \log x + C$ when K and C are constants. Further with Bernoulli let us assume that, a being the income which affords the *barest* necessaries of life, pain exceeds pleasure when the income is less than a, and balances it when the income equals a; then our equation becomes $y = K \log \dfrac{x}{a}$. Of course both K and a vary with the temperament, the health, the habits, and the social surroundings of each individual. Laplace gives to x the name *fortune physique*, and to y the name *fortune morale*.

Bernoulli himself seems to have thought of x and a as representing certain amounts of property rather than of income; but we cannot estimate the property necessary for life without some understanding as to the length of time during which it is to support life, that is, without really treating it as income.

Perhaps the guess which has attracted most attention after Bernoulli's is Cramer's suggestion that the pleasure afforded by wealth may be taken to vary as the square root of its amount.

NOTE IX. (p. 135). The argument that fair gambling is an economic blunder is generally based on Bernoulli's or some other definite hypothesis. But it requires no further assumption than that, firstly the pleasure of gambling may be neglected; and, secondly $\phi''(x)$ is negative for all values of x, where $\phi(x)$ is the pleasure derived from wealth equal to x.

For suppose that the chance that a particular event will happen is p, and a man makes a fair bet of py against $(1 - p)y$ that it will happen. By so doing he changes his expectation of happiness from

$$\phi(x) \text{ to } p\phi\{x+(1-p)y\}+(1-p)\phi(x-py).$$

This when expanded by Taylor's Theorem becomes

$$\begin{aligned}\phi(x) \quad &+\frac{1}{2}p(1-p)^2y^2\phi''\{x+\theta(1-p)y\}\\ &+\frac{1}{2}p^2(1-p)y^2\phi''(x-\Theta py);\end{aligned}$$

assuming $\phi''(x)$ to be negative for all values of x, this is always less than $\phi(x)$.

It is true that this loss of probable happiness need not be greater than the pleasure derived from the excitement of gambling, and we are then thrown back upon the induction that pleasures of gambling are in Bentham's phrase "impure"; since experience shows that they are likely to engender a restless, feverish character, unsuited for steady work as well as for the higher and more solid pleasures of life.

NOTE X. (p. 142). Following on the same lines as in Note I., let us take v to represent the disutility or discommodity of an amount of labour l, then $\frac{dv}{dl}$ measures the marginal degree of disutility of labour; and, subject to the qualifications mentioned in the text, $\frac{d^2v}{dl^2}$ is positive.

Let m be the amount of money or general purchasing power at a person's disposal, μ its total utility to him, and therefore $\frac{d\mu}{dm}$ its marginal utility. Thus if Δw be the wages that must be paid him to induce him to do labour Δl, then $\Delta w\frac{d\mu}{dm}=\Delta v$, and

$$\frac{dw}{dl} \cdot \frac{d\mu}{dm} = \frac{dv}{dl}.$$

If we assume that his dislike to labour is not a fixed, but a fluctuating quantity, we may regard $\dfrac{dw}{dl}$ as a function of m, v, and l; and then both $\dfrac{d^2w}{dm\, dl}$, $\dfrac{d^2w}{dv\, dl}$ are always positive.

NOTE XI. (p. 248). If members of any species of bird begin to adopt aquatic habits, every increase in the webs between the toes—whether coming about gradually by the operation of natural selection, or suddenly as a sport,—will cause them to find their advantage more in aquatic life, and will make their chance of leaving offspring depend more on the increase of the web. So that, if $f(t)$ be the average area of the web at time t, then the rate of increase of the web increases (within certain limits) with every increase in the web, and therefore $f''(t)$ is positive. Now we know by Taylor's Theorem that

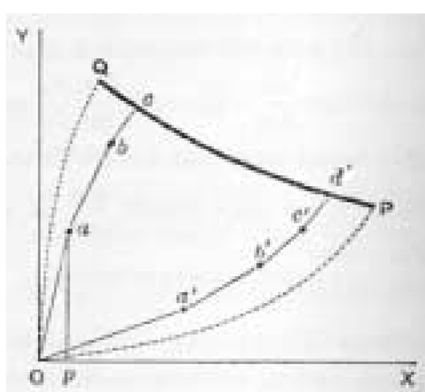
$$f(t+h) = f(t) + hf'(t) + \frac{h^2}{1.2} f''(t+\theta h);$$

and if h be large, so that h^2 is very large, then $f(t+h)$ will be much greater than $f(t)$ even though $f'(t)$ be small and $f''(t)$ is never large. There is more than a superficial connection between the advance made by the applications of the differential calculus to physics at the end of the eighteenth century and the beginning of the nineteenth, and the rise of the theory of evolution. In sociology as well as in biology we are learning to watch the accumulated effects of forces which, though weak at first, get greater strength from the growth of their own effects; and the universal form, of which every such fact is a special embodiment, is Taylor's Theorem; or, if the action of more than one cause at a time is to be taken account of, the corresponding expression of a function of several variables. This conclusion will remain valid even if further investigation confirms the suggestion, made by some Mendelians, that gradual changes in the race are originated by large divergences of individuals from the prevailing type. For economics is a study of mankind of particular nations, of particular social strata; and it is only indirectly concerned with the lives of men of exceptional genius or exceptional wickedness and violence.

NOTE XII. (p. 331). If, as in Note X., v be the discommodity of the amount of labour which a person has to exert in order to obtain an amount x of a commodity from which he derives a pleasure u, then the pleasure of having further supplies will be equal to the pain of getting them when $\frac{du}{dx} = \frac{dv}{dx}$.

If the pain of labour be regarded as a negative pleasure; and we write $U \equiv -v$; then $\frac{du}{dx} + \frac{du}{dx} = 0$, i.e. $u + U =$ a maximum at the point at which his labour ceases.

NOTE XII. *bis* (p. 793). In an article in the *Giornale degli Economisti* for February, 1891, Prof. Edgeworth draws the adjoining diagram, which represents the cases of barter of apples for nuts described on pp. 414–6. Apples are measured along Ox, and nuts along Oy; $Op = 4$, $pa = 40$; and a represents the termination of the first bargain in which 4 apples have been exchanged for



40 nuts, in the case in which A gets the advantage at starting: b represents the second, and c the final stage of that case. On the other hand, a' represents the first, and b', c', d' the second, third, and final stages of the set of bargains in which B gets the advantage at starting. QP, the locus on which c and d' must both necessarily lie, is called by Prof. Edgeworth the *Contract Curve*.

Following a method adopted in his *Mathematical Psychics* (1881), he takes U to represent the total utility to A of apples and nuts when he has given up x apples and received y nuts, V the total utility to B of apples and nuts when he has received x apples and given up y nuts. If an additional Δx apples are exchanged for Δy nuts, the exchange will be indifferent to A if

$$\frac{dU}{dx}\Delta x + \frac{dU}{dy}\Delta y = 0;$$

and it will be indifferent to B if $\dfrac{dV}{dx}\Delta x + \dfrac{dV}{dy}\Delta y = 0$. These, therefore, are the equations to the indifference curves OP and OQ of the figure respectively; and the contract curve which is the locus of points at which the terms of exchange that are indifferent to A are also indifferent to B has the elegant equation $\dfrac{dU}{dx} \div \dfrac{dU}{dy} = \dfrac{dV}{dx} \div \dfrac{dV}{dy}$.

If the marginal utility of nuts be constant for A and also for B, $\dfrac{dU}{dy}$ and $\dfrac{dV}{dy}$ become constant; U becomes $\Phi(a - x) + \alpha y$, and V becomes $\Psi(a - x) + \beta y$; and the contract curve becomes $F(x) = 0$; or $x = C$; that is, it is a straight line parallel to Oy, and the value of $\Delta y : \Delta x$ given by either of the indifference curves, a function of C; thus showing that by whatever route the barter may have started, equilibrium will have been found at a point at which C apples have been exchanged, and the final rate of exchange is a function of C; that is, it is a constant also. This last application of Prof. Edgeworth's mathematical version of the theory of barter, to confirm the results reached in the text, was first made by Mr Berry, and is published in the *Giornale degli Economisti* for June, 1891.

Prof. Edgeworth's plan of representing U and V as general functions of x and y has great attractions to the mathematician; but it seems less adapted to express the every-day facts of economic life than that of regarding, as Jevons did, the marginal utilities of apples as functions of x simply. In that case, if A had no nuts at starting, as is assumed in the particular case under discussion, U takes the form

$$\int_0^x \phi_1(a-x)dx + \int_0^y \psi_1(y)dy;$$

similarly for V. And then the equation to the contract curve is of the form

$$\phi_1(a-x) \div \psi_1(y) = \phi_2(x) \div \psi_2(b-y);$$

which is one of the Equations of Exchange in Jevon's *Theory*, 2nd Edition, p. 108.

NOTE XIII. (p. 354). Using the same notation as in Note V., let us take our starting-point as regards time at the date of beginning to build the house, and let T' be the time occupied in building it. Then the present value of the pleasures, which he expects to derive from the house, is

$$H = \int_{T'}^{T} \varpi R^{-t} \frac{dh}{dt} dt.$$

Let Δv be the element of effort that will be incurred by him in building the house in the interval of time Δt (between the time t and the time $t + \Delta t$), then the present value of the aggregate of effort is

$$V = \int_0^{T'} R^{-t} \frac{dv}{dt} dt.$$

If there is any uncertainty as to the labour that will be required, every possible element must be counted in, multiplied by the probability, ϖ', of its being required; and then V becomes $\int_0^{T'} \varpi R^{-t} \frac{dv}{dt} dt.$

If we transfer the starting-point to the date of the completion of the house, we have

$$H = \int_0^{T_1} \varpi R^{-t} \frac{dh}{dt} dt \quad \text{and} \quad V = \int_0^{T'} \varpi R^{t} \frac{dv}{dt} dt,$$

where $T_1 = T - T'$; and this starting-point, though perhaps the less natural from the mathematical point of view, is the more natural from the point of view of ordinary business. Adopting it, we see V as the aggregate of estimated pains incurred; each bearing on its back, as it were, the accumulated burden of the waitings between the time of its being incurred and the time when it begins to bear fruit.

Jevons' discussion of the investment of capital is somewhat injured by the unnecessary assumption that the function representing it is an expression of the first order; which is the more remarkable as he had himself, when describing Gossen's work, pointed out the objections to the plan followed by him (and Whewell) of substituting straight lines for the multiform curves that represent the true characters of the variations of economic quantities.

NOTE XIV. (p. 357). Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be the several amounts of different kinds of labour, as, for instance, wood-cutting, stone-carrying, earth-digging, etc., on the part of the man in question that would be used in building the house on any given plan; and β, β', β'', etc., the several amounts of accommodation of different kinds such as sitting-rooms, bed-rooms, offices, etc. which the house would afford on that plan. Then, using V and H as in the previous note, $V, \beta, \beta', \beta''$ are all functions of $\alpha_1, \alpha_2, \alpha_3, \ldots$, and H being a function of $\beta, \beta', \beta'', \ldots$ is a function also of $\alpha_1, \alpha_2, \alpha_3, \ldots$ We have, then, to find the marginal investments of each kind of labour for each kind of use.

$$\begin{aligned}
\frac{dV}{da_1} &= \frac{dH}{d\beta}\frac{d\beta}{da_1} = \frac{dH}{d\beta'}\frac{d\beta'}{da_1} = \frac{dH}{d\beta''}\frac{d\beta''}{da_1} = - \cdots \\
\frac{dV}{da_2} &= \frac{dH}{d\beta}\frac{d\beta}{da_2} = \frac{dH}{d\beta'}\frac{d\beta'}{da_2} = \frac{dH}{d\beta''}\frac{d\beta''}{da_2} = \cdots .
\end{aligned}$$

These equations represent a balance of effort and benefit. The real cost to him of a little extra labour spent on cutting timber and working it up is just balanced by the benefit of the extra sitting-room or bed-room accommodation that he could get by so doing. If, however, instead of doing the work himself, he pays carpenters to do it, we must take V to represent, not his total effort, but his total outlay of general purchasing power. Then the rate of pay which he is willing to give to carpenters for further labour, his marginal demand price for their labour, is measured by $\frac{dV}{da}$; while $\frac{dH}{d\beta}, \frac{dH}{d\beta'}$ are the money measures to him of the marginal utilities of extra sitting-room and bed-room accommodation respectively, that is, his marginal demand prices for them; and $\frac{d\beta}{da}, \frac{d\beta'}{da}$ are the marginal efficiencies of carpenters' labour in providing those accommodations. The equations then state that the demand price for carpenters' labour tends to be equal to the demand price for extra sitting-room accommodation, and also for extra bed-room accommodation and so on, multiplied in each case by the marginal efficiency of the work of carpenters in providing that extra accommodation, proper units being chosen for each element.

When this statement is generalized, so as to cover all the varied demand in a market for carpenters' labour, it becomes:—the (marginal) demand price for carpenters' labour is the (marginal) efficiency of carpenters' labour in increasing the supply of any product, multiplied by the (marginal) demand price for that product. Or, to put the same thing in other words, the wages of a unit of carpenters' labour tends to be equal to the value of such part of any of the products, to producing which their labour contributes, as represents the marginal efficiency of a unit of carpenters' labour with regard to that product; or, to use a phrase, with which we shall be much occupied in Book VI ch. I., it tends to be equal to the value of the "net product" of their labour. This proposition is very important and contains within itself the kernel of the demand side of the theory of distribution.

Let us then suppose a master builder to have it in mind to erect certain buildings, and to be considering what different accommodation he shall provide; as, for instance, dwelling-houses, warehouses, factories, and retail shop-room. There will be two classes of questions for him to decide: how much accommodation of each kind he shall provide, and by what means he shall provide it. Thus, besides deciding whether to erect villa residences, offering a certain amount of accommodation, he has to decide what agents of production he will use, and in what proportions: whether e.g. he will use tile or slate; how much stone he will use; whether he will use steam power for making mortar etc. or only for crane work; and, if he is in a large town, whether he will have his scaffolding put up by men who make that work a speciality or by ordinary labourers; and so on.

Let him then decide to provide an amount β of villa accommodation, an amount β' of warehouse, an amount β'' of factory accommodation, and so on, each of a certain class. But, instead of supposing him to hire simply $\alpha_1, \alpha_2, \ldots$ quantities of different kinds of labour, as before, let us class his expenditure, under the three heads of (1) wages, (2) prices of raw material, and (3) interest on capital: while the value of his own work and enterprise makes a fourth head.

Thus let $x_1, x_2, \ldots$ be the amounts of different classes of labour, including that of superintendence, which he hires; the amount of each kind of labour being made up of its duration and its intensity.

Let $y_1, y_2, \ldots$ be amounts of various kinds of raw materials, which are used up and embodied in the buildings; which may be supposed to be sold freehold. In that case, the pieces of land on which they are severally built are merely particular forms of raw material from the present point of view, which is that of the individual undertaker.

Next let z be the amount of locking up, or appropriation of the employment, of capital for the several purposes. Here we must reckon in all forms of capital reduced to a common money measure, including advances for wages, for the purchase of raw material; also the uses, allowing for wear-and-tear etc. of his plant of all kinds: his workshops themselves and the ground on which they are built being reckoned on the same footing. The periods, during which the various lockings up run, will vary; but they must be reduced, on a "compound rate," i.e. according to geometrical progression, to a standard unit, say a year.

Fourthly, let u represent the money equivalent of his own labour, worry, anxiety, wear-and-tear etc. involved in the several undertakings.

In addition, there are several elements, which might have been entered under separate heads; but which we may suppose combined with those already mentioned. Thus the allowance to be made for risk may be shared between the last two heads. A proper share of the general expenses of working the business ("supplementary costs," see p. 360) will be distributed among the four heads of wages, raw materials, interest on the capital value of the organization of the business (its goodwill etc.) regarded as a going concern, and remuneration of the builder's own work, enterprise and anxiety.

Under these circumstances V represents his total outlay, and H his total receipts; and his efforts are directed to making $H - V$ a maximum. On this plan, we have equations similar to those already given, viz.:—

$$\frac{dV}{dx_1} = \frac{dH}{d\beta} \cdot \frac{d\beta}{dx_1} = \frac{dH}{d\beta'} \cdot \frac{d\beta'}{dx_1} = \dots$$

$$\frac{dV}{dx_2} = \frac{dH}{d\beta} \cdot \frac{d\beta}{dx_2} = \frac{dH}{d\beta'} \cdot \frac{d\beta'}{dx_2} = \dots$$

$$\dots\dots$$

$$\frac{dV}{dy_1} = \frac{dH}{d\beta} \cdot \frac{d\beta}{dy_1} = \frac{dH}{d\beta'} \cdot \frac{d\beta'}{dy_1} = \dots$$

$$\dots\dots$$

$$\frac{dV}{dz} = \frac{dH}{d\beta} \cdot \frac{d\beta}{dz} = \frac{dH}{d\beta'} \cdot \frac{d\beta'}{dz} = \dots$$
$$\frac{dV}{du} = \frac{dH}{d\beta} \cdot \frac{d\beta}{du} = \frac{dH}{d\beta'} \cdot \frac{d\beta'}{du} = \dots.$$

That is to say, the marginal outlay which the builder is willing to make for an additional small supply, δx_1, of the first class of labour, viz. $\frac{dV}{dx_1}\delta x_1$, is equal to $\frac{dH}{d\beta} \cdot \frac{d\beta}{dx_1}\delta x_1$; i.e. to that increment in his total receipts H, which he will obtain by the increase in the villa accommodation provided by him that will result from the extra small supply of the first class of labour: this will equal a similar sum with regard to warehouse accommodation, and so on. Thus he will have distributed his resources between various uses in such a way that he would gain nothing by diverting any part of any agent of production—labour, raw material, the use of capital—nor his own labour and enterprise from one class of building to another: also he would gain nothing by substituting one agent for another in any branch of his enterprise, nor indeed by any increase or diminution of his use of any agent. From this point of view our equations have a drift very similar to the argument of Book III. ch. V. as to the choice between the different uses of the same thing. (Compare one of the most interesting notes (f) attached to Prof. Edgeworth’s brilliant address to the British Association in 1889.)

There is more to be said (see V. XI. 1, and VI. I. 8) on the difficulty of interpreting the phrase the “net product" of any agent of production, whether a particular kind of labour or any other agent; and perhaps the rest of this note, though akin to what has gone before, may more conveniently be read at a later stage. The builder paid $\frac{dV}{dx_1}\delta x_1$ for the last element of the labour of the first group because that was its net product; and, if directed to building villas, it brought him in $\frac{dH}{d\beta}\cdot\frac{d\beta}{dx_1}\delta x_1$, as special receipts. Now if p be the price per unit, which he receives for an amount β of villa accommodation, and therefore $p\beta$ the price which he receives for the whole amount β; and if we put for shortness $\Delta\beta$ in place of $\frac{d\beta}{dx_1}\delta x_1$, the increase of villa accommodation due to the additional element of labour δx_1; then the net product we are seeking is not $p\Delta\beta$, but $p\Delta\beta + \beta\Delta p$; where Δp is a negative quantity, and is the fall in demand price caused by the increase in the amount of villa accommodation offered by the builder. We have to make some study of the relative magnitudes of these two elements $p\Delta\beta$ and $\beta\Delta p$.

If the builder monopolized the supply of villas, β would represent the total supply of them: and, if it happened that the elasticity of the demand for them was less than unity, when the amount β was offered, then, by increasing his supply, he would diminish his total receipts; and $p\Delta\beta+\beta\Delta p$ would be a negative quantity. But of course he would not have allowed the production to go just up to an amount at which the demand would be thus inelastic. The margin which he chose for his production would certainly be one for which the negative quantity $\beta\Delta p$ is less than $p\Delta\beta$, but not necessarily so much less that it may be neglected in comparison. This is a dominant fact in the theory of monopolies discussed in Book V. chapter XIV.

It is dominant also in the case of any producer who has a limited trade connection which he cannot quickly enlarge. If his customers have already as much of his wares as they care for, so that the elasticity of their demand is temporarily less than unity, he might lose by putting on an additional man to work for him, even though that man would work for nothing. This fear of temporarily spoiling a man's special market is a leading influence in many problems of value relating to short periods (see Book V. chs. V. VII. XI.); and especially in those periods of commercial depression, and in those regulations of trade associations, formal and informal, which we shall have to study in the second volume. There is an allied difficulty in the case of commodities of which the expenses of production diminish rapidly with every increase in the amount produced: but here the causes that govern the limits of production are so complex that it seems hardly worth while to attempt to translate them into mathematical language. See V. XII. 2.

When however we are studying the action of an individual undertaker with a view of illustrating the normal action of the causes which govern the general demand for the several agents of production, it seems clear that we should avoid cases of this kind. We should leave their peculiar features to be analysed separately in special discussions, and take our normal illustration from a case in which the individual is only one of many who have efficient, if indirect, access to the market. If $\beta \Delta p$ be numerically equal to $p \Delta \beta$, where β is the whole production in a large market; and an individual undertaker produced β', a thousandth part of β; then the increased receipt from putting on an additional man is $p \Delta \beta'$, which is the same as $p \Delta \beta$; and the deduction to be made from it is only $\beta' \Delta p$, which is a thousandth part of $\beta \Delta p$ and may be neglected. For the purpose therefore of illustrating a part of the general action of the laws of distribution we are justified in speaking of the value of the net product of the marginal work of any agent of production as the amount of that net product taken at the normal selling value of the product, that is, as $p\Delta\beta$.

It may be noticed that none of these difficulties are dependent upon the system of division of labour and work for payment; though they are brought into prominence by the habit of measuring efforts and satisfactions by price, which is associated with it. Robinson Crusoe erecting a building for himself would not find that an addition of a thousandth part to his previous accommodation increased his comfort by a thousandth part. What he added might be of the same character with the rest; but if one counted it in at the same rate of real value to him, one would have to reckon for the fact that the new part made the old of somewhat less urgent need, of somewhat lower real value to him (see note 1 on p. 417). On the other hand, the law of increasing return might render it very difficult for him to assign its true net product to a given half-hour's work. For instance, suppose that some small herbs, grateful as condiment, and easily portable, grow in a part of his island, which it takes half a day to visit; and he has gone there to get small batches at a time. Afterwards he gives a whole day, having no important use to which he can put less than half a day, and comes back with ten times as great a load as before. We cannot then separate the return of the last half-hour from the rest; our only plan is to take the whole day as a unit, and compare its return of satisfaction with those of days spent in other ways; and in the modern system of industry we have the similar, but more difficult task of taking, for some purposes, the whole of a process of production as a single unit.

It would be possible to extend the scope of such systems of equations as we have been considering, and to increase their detail, until they embraced within themselves the whole of the demand side of the problem of distribution. But while a mathematical illustration of the mode of action of a definite set of causes may be complete in itself, and strictly accurate within its clearly defined limits, it is otherwise with any attempt to grasp the whole of a complex problem of real life, or even any considerable part of it, in a series of equations. For many important considerations, especially those connected with the manifold influences of the element of time, do not lend themselves easily to mathematical expression: they must either be omitted altogether, or clipped and pruned till they resemble the conventional birds and animals of decorative art. And hence arises a tendency towards assigning wrong proportions to economic forces; those elements being most emphasized which lend themselves most easily to analytical methods. No doubt this danger is inherent in every application not only of mathematical analysis, but of analysis of any kind, to the problems of real life. It is a danger which more than any other the economist must have in mind at every turn. But to avoid it altogether, would be to abandon the chief means of scientific progress: and in discussions written specially for mathematical readers it is no doubt right to be very bold in the search for wide generalizations.

In such discussions it may be right, for instance, to regard H as the sum total of satisfactions, and V as the sum total of dissatisfactions (efforts, sacrifices etc.) which accrue to a community from economic causes; to simplify the notion of the action of these causes by assumptions similar to those which are involved, more or less consciously, in the various forms of the doctrine that the constant drift of these causes is towards the attainment of the *Maximum Satisfaction*, in the net aggregate for the community (see above pp. 470–5); or, in other words, that there is a constant tendency to make $H - V$ a maximum for society as a whole. On this plan the resulting differential equations of the same class as those which we have been discussing, will be interpreted to represent value as governed in every field of economics by the balancing of groups of utilities against groups of disutilities, of groups of satisfactions against groups of real costs. Such discussions have their place: but it is not in a treatise such as the present, in which mathematics are used only to express in terse and more precise language those methods of analysis and reasoning which ordinary people adopt, more or less consciously, in the affairs of every-day life.

It may indeed be admitted that such discussions have some points of resemblance to the method of analysis applied in Book III. to the total utility of particular commodities. The difference between the two cases is mainly one of degree: but it is of a degree so great as practically to amount to a difference of kind. For in the former case we take each commodity by itself and with reference to a particular market; and we take careful account of the circumstances of the consumers at the time and place under consideration. Thus we follow, though perhaps with more careful precautions, the practice of ministers of finance, and of the common man when discussing financial policy. We note that a few commodities are consumed mainly by the rich; and that in consequence their real total utilities are less than is suggested by the money measures of those utilities. But we assume, with the rest of the world, that as a rule, and in the absence of special causes to the contrary, the real total utilities of two commodities that are mainly consumed by the rich stand to one another in about the same relation as their money measures do: and that the same is true of commodities the consumption of which is divided out among rich and middle classes and poor in similar proportions. Such estimates are but rough approximations; but each particular difficulty, each source of possible error, is pushed into prominence by the definiteness of our phrases: we introduce no assumptions that are not latent in the practice of ordinary life; while we attempt no task that is not grappled with in a rougher fashion, but yet to good purpose, in the practice of ordinary life: we introduce no new assumptions, and we bring into clear light those which cannot be avoided. But though this is possible when dealing with particular commodities with reference to particular markets, it does not seem possible with regard to the innumerable economic elements that come within the all-embracing net of the doctrine of Maximum Satisfaction. The forces of supply are especially heterogeneous and complex: they include an infinite variety of efforts and sacrifices, direct and indirect, on the part of people in all varieties of industrial grades: and if there were no other hindrance to giving a concrete interpretation to the doctrine, a fatal obstacle would be found in its latent assumption that the cost of rearing children and preparing them for their work can be measured in the same way as the cost of erecting a machine.

For reasons similar to those given in this typical case, our mathematical notes will cover less and less ground as the complexity of the subjects discussed in the text increases. A few of those that follow relate to monopolies, which present some sides singularly open to direct analytical treatment. But the majority of the remainder will be occupied with illustrations of joint and composite demand and supply which have much in common with the substance of this note: while the last of that series Note XXI. goes a little way towards a general survey of the problem of distribution and exchange (without reference to the element of time), but only so far as to make sure that the mathematical illustrations used point towards a system of equations, which are neither more nor less in number than the unknowns introduced into them.

NOTE XIV. *bis* (p. 384). In the diagrams of this chapter (V. VI.) the supply curves are all inclined positively; and in our mathematical versions of them we shall suppose the marginal expenses of production to be determined with a definiteness that does not exist in real life: we shall take no account of the time required for developing a representative business with the internal and external economies of production on a large scale; and we shall ignore all those difficulties connected with the law of increasing return which are discussed in Book V. ch. XII. To adopt any other course would lead us to mathematical complexities, which though perhaps not without their use, would be unsuitable for a treatise of this kind. The discussions therefore in this and the following notes must be regarded as sketches rather than complete studies.

Let the factors of production of a commodity A be a_1, a_2, etc.; and let their supply equations be $y = \phi_1(x)$, $y = \phi_2(x)$, etc. Let the number of units of them required for the production of x units of A be $m_1x, m_2x, \ldots$ respectively; where $m_1, m_2, \ldots$ are generally not constants but functions of x. Then the supply equation of A is

$$y = \Phi(x) = m_1\phi_1(m_1x) + m_2\phi_2(m_2x) + \cdots \equiv \Sigma\{m\phi(mx)\}.$$

Let $y = F(x)$ be the demand equation for the finished commodity, then the derived demand equation for a_r the r^{th} factor is

$$y = F(x) - \{\phi(x) - m_r\phi_r(m_rx)\}.$$

But in this equation y is the price, not of one unit of the factor but of m units; and to get an equation expressed in terms of fixed units let η be the price of one unit, and let $\xi = m_r x$, then $n = \frac{1}{m_r} \cdot y$ and the equation becomes

$$\eta = f_r(\xi) = \frac{1}{m_r}\left[F\left(\frac{1}{m_r}\xi\right) - \left\{\phi\left(\frac{1}{m_r}\xi\right) - m_r\phi_r(\xi)\right\}\right].$$

If m_r is a function of x, say $= \psi_r(x)$; then x must be determined in terms of ξ by the equation $\xi = x\psi_r(x)$, so that m_r can be written $\chi_r(\xi)$; substituting this we have η expressed as a function of ξ. The supply equation for a_r is simply $\eta = \phi_r(\xi)$.

NOTE XV. (p. 386). Let the demand equation for knives be

$$y = F(x) \text{.................}(1),$$

let the supply equation for knives be $y = \Phi(x)$.................(2),
let that for handles be $y = \phi_1(x)$ ....(3),
and that for blades be $y = \phi_2(x)$ ....(4),

then the demand equation for handles is

$$y = f_1(x) = F(x) - \phi_2(x) \text{ ......}(5).$$

The measure of elasticity for (5) is $-\left\{\frac{xf_1'(x)}{f_1(x)}\right\}^{-1}$, that is,

$$-\left\{\frac{xF'(x) - x\phi_2'(x)}{f_1(x)}\right\}^{-1};$$

that is,
$$\left\{-\frac{xF'(x)}{F(x)} \cdot \frac{F(x)}{f_1(x)} + \frac{x\phi_2'(x)}{f_1(x)}\right\}^{-1}.$$

This will be the smaller the more fully the following conditions are satisfied: (i) that $-\frac{xF'(x)}{F(x)}$, which is necessarily positive, be large, i.e. that the elasticity of the demand for knives be small; (ii) that $\phi_2'(x)$ be positive and large, i.e. that the supply price for blades should increase rapidly with an increase, and diminish rapidly with a diminution of the amount supplied; and (iii) that $\frac{F(x)}{f_1(x)}$ should be large; that is, that the price of handles should be but a small part of the price of knives.

A similar, but more complex inquiry, leads to substantially the same results, when the units of the factors of production are not fixed, but vary as in the preceding note.

NOTE XVI. (p. 387). Suppose that m bushels of hops are used in making a gallon of ale of a certain kind, of which in equilibrium x' gallons are sold at a price $y' = F(x')$. Let m be changed into $m + \Delta m$; and, as a result, when x' gallons are still offered for sale let them find purchasers at a price $y' + \Delta y'$; then $\frac{\Delta y'}{\Delta m}$ represents the marginal demand price for hops: if it is greater than their supply price, it will be to the interest of the brewers to put more hops into the ale. Or, to put the case more generally, let $y = F(x, m)$, $y = \Phi(x, m)$ be the demand and supply equations for beer, x being the number of gallons and m the number of bushels of hops in each gallon. Then $F(x, m) - \Phi(x, m) =$ excess of demand over supply price. In equilibrium this is of course zero: but if it were possible to make it a positive sum by varying m the change would be effected: therefore (assuming that there is no perceptible change in the expense of making the beer, other than what results from the increased amount of hops) $\frac{dF}{dm} = \frac{d\Phi}{dm}$: the first represents the marginal demand price, and the second the marginal supply price of hops; and these two are therefore equal.

This method is of course capable of being extended to cases in which there are concurrent variations in two or more factors of production.

NOTE XVII. (p. 388). Suppose that a thing, whether a finished commodity or a factor of production, is distributed between two uses, so that of the total amount x the part devoted to the first use is x_1, and that devoted to the second use is x_2. Let $y = \phi(x)$ be the total supply equation; $y = f_1(x_1)$ and $y = f_2(x_2)$ be the demand equations for its first and second uses. Then in equilibrium the three unknowns x, x_1, and x_2 are determined by the three equations $f_1(x_1) = f_2(x_2) = \phi(x)$; $x_1 + x_2 = x$.

Next suppose that it is desired to obtain separately the relations of demand and supply of the thing in its first use, on the supposition that, whatever perturbations there may be in its first use, its demand and supply for the second use remains in equilibrium; i.e. that its demand price for the second use is equal to its supply price for the total amount that is actually produced, i.e. $f_2(x_2) = \phi(x_1 + x_2)$ always. From this equation we can determine x_2 in terms of x_1, and therefore x in terms of x_1; and therefore we can write $\phi(x) = \psi(x_1)$. Thus the supply equation for the thing in its first use becomes $y = \psi(x_1)$; and this with the already known equation $y = f_1(x_1)$ gives the relations required.

NOTE XVIII. (p. 389). Let $a_1, a_2, \ldots$ be joint products, $m_1x, m_2x, \ldots$ of them severally being produced as the result of x units of their joint process of production, for which the supply equation is $y = \phi(x)$. Let

$$y = f_1(x), \quad y = f_2(x), \ldots$$

be their respective demand equations. Then in equilibrium

$$m_1 f_1(m_1 x) + m_2 f_2(m_2 x) + \cdots = \phi(x).$$

Let x' be the value of x determined from this equation; then $f_1(m_1 x')$, $f_2(m_2 x')$ etc. are the equilibrium prices of the several joint products. Of course m_1, m_2 are expressed if necessary in terms of x'.

NOTE XIX. (p. 390). This case corresponds, *mutatis mutandis*, to that discussed in Note XVI. If in equilibrium x' oxen annually are supplied and sold at a price $y' = \phi(x')$; and each ox yields m units of beef: and if breeders find that by modifying the breeding and feeding of oxen they can increase their meat-yielding properties to the extent of Δm units of beef (the hides and other joint products being, on the balance, unaltered), and that the extra expense of doing this is $\Delta y'$, then $\dfrac{\Delta y'}{\Delta m}$ represents the marginal supply price of beef: if this price were less than the selling price, it would be to the interest of breeders to make the change.

NOTE XX. (p. 391). Let $a_1, a_2, \ldots$ be things which are fitted to subserve exactly the same function. Let their units be so chosen that a unit of any one of them is equivalent to a unit of any others. Let their several supply equations be $y_1 = \phi_1(x_1),\ y_2 = \phi_2(x_2), \ldots.$

In these equations let the variable be changed, and let them be written $x_1 = \psi_1(y_1),\ x_2 = \psi_2(y_2), \ldots$. Let $y = f(x)$ be the demand equation for the service for which all of them are fitted. Then in equilibrium x and y are determined by the equations $y = f(x);\ x = x_1 + x_2 + \cdots,\ y_1 = y_2 = \cdots = y$. (The equations must be such that none of the quantities $x_1, x_2, \ldots$ can have a negative value. When y_1 has fallen to a certain level x_1 becomes zero; and for lower values x_1 remains zero; it does not become negative.) As was observed in the text, it must be assumed that the supply equations all conform to the law of diminishing return; i.e. that $\phi_1'(x), \phi_2'(x), \ldots$ are always positive.

NOTE XXI. (p. 393). We may now take a bird's-eye view of the problems of joint demand, composite demand, joint supply and composite supply when they all arise together, with the object of making sure that our abstract theory has just as many equations as it has unknowns, neither more nor less.

In a problem of joint demand we may suppose that there are n commodities $A_1, A_2, \ldots A_n$. Let A_1 have a_1 factors of production, let A_2 have a_2 factors, and so on, so that the total number of factors of production is $a_1 + a_2 + a_3 + \cdots + a_n$: let this $= m$.

First, suppose that all the factors are different, so that there is no composite demand; that each factor has a separate process of production, so that there are no joint products; and lastly, that no two factors subserve the same use, so that there is no composite supply. We then have $2n + 2m$ unknowns, viz. the amounts and prices of n commodities and of m factors; and to determine them we have $2m + 2n$ equations, viz.—(i) n demand equations, each of which connects the price and amount of a commodity; (ii) n equations, each of which equates the supply price for any amount of a commodity to the sum of the prices of corresponding amounts of its factors; (iii) m supply equations, each of which connects the price of a factor with its amount; and lastly, (iv) m equations, each of which states the amount of a factor which is used in the production of a given amount of the commodity.

Next, let us take account not only of joint demand but also of composite demand. Let β_1 of the factors of production consist of the same thing, say carpenters' work of a certain efficiency; in other words, let carpenters' work be one of the factors of production of β_1 of the n commodities $A_1, A_2, \ldots$. Then since the carpenters' work is taken to have the same price in whatever production it is used, there is only one price for each of these factors of production, and the number of unknowns is diminished by $\beta_1 - 1$; also the number of supply equations is diminished by $\beta_1 - 1$: and so on for other cases.

Next, let us in addition take account of joint supply. Let γ_1 of the things used in producing the commodities be joint products of one and the same process. Then the number of unknowns is not altered; but the number of supply equations is reduced by $(\gamma_1 - 1)$: this deficiency is however made up by a new set of $(\gamma_1 - 1)$ equations connecting the amounts of these joint products: and so on.

Lastly, let one of the things used have a composite supply made up from δ_1 rival sources: then, reserving the old supply equations for the first of these rivals, we have $2(\delta_1 - 1)$ additional unknowns, consisting of the prices and amounts of the remaining $(\delta_1 - 1)$ rivals. These are covered by $(\delta_1 - 1)$ supply equations for the rivals and $(\delta_1 - 1)$ equations between the prices of the δ_1 rivals.

Thus, however complex the problem may become, we can see that it is theoretically determinate, because the number of unknowns is always exactly equal to the number of the equations which we obtain.

NOTE XXII. (p. 480). If $y = f_1(x),\ y = f_2(x)$ be the equations to the demand and supply curves respectively, the amount of production which affords the maximum monopoly revenue is found by making $\{xf_1(x) - xf_2(x)\}$ a maximum; that is, it is the root, or one of the roots of the equation

$$\frac{d}{dx}\{xf_1(x) - xf_2(x)\} = 0.$$

The supply function is represented here by $f_2(x)$ instead of as before by $\phi(x)$, partly to emphasize the fact that supply price does not mean exactly the same thing here as it did in the previous notes, partly to fall in with that system of numbering the curves which is wanted to prevent confusion now that their number is being increased.

NOTE XXIII. (p. 482). If a tax be imposed of which the aggregate amount is $F(x)$, then, in order to find the value of x which makes the monopoly revenue a maximum, we have $\frac{d}{dx}\{xf_1(x) - xf_2(x) - F(x)\} = 0$; and it is clear that if $F(x)$ is either constant, as in the case of a license duty, or varies as $xf_1(x) - xf_2(x)$, as in the case of an income-tax, this equation has the same roots as it would have if $F(x)$ were zero.

Treating the problems geometrically, we notice that, if a fixed burden be imposed on a monopoly sufficiently to make the monopoly revenue curve fall altogether below Ox, and q' be the point on the new curve vertically below L in fig. 35, then the new curve at q' will touch one of a series of rectangular hyperbolas drawn with yO produced downwards for one asymptote and Ox for the other. These curves may be called Constant Loss curves.

Again, a tax proportionate to the monopoly revenue, and say

m times that revenue (m being less than 1), will substitute for QQ' a curve each ordinate of which is $(1 - m)\times$ the ordinate of the corresponding point on QQ'; i.e. the point which has the same abscissa. The tangents to corresponding points on the old and new positions of QQ' will cut Ox in the same point, as is obvious by the method of projections. But it is a law of rectangular hyperbolas which have the same asymptotes that, if a line be drawn parallel to one asymptote to cut the hyperbolas, and tangents be drawn to them at its points of intersection, they will all cut the other asymptote in the same point. Therefore if q_3' be the point on the new position of QQ' corresponding to q_3, and if we call G the point in which the common tangent to the hyperbola and QQ' cuts Ox, Gq_3' will be a tangent to the hyperbola which passes through q_3'; that is, q_3' is a point of maximum revenue on the new curve.

The geometrical and analytical methods of this note can be applied to cases, such as are discussed in the latter part of §4 in the text, in which the tax is levied on the produce of the monopoly.

NOTE XXIII. *bis* (p. 489). These results have easy geometrical proofs by Newton's method, and by the use of well-known properties of the rectangular hyperbola. They may also be proved analytically. As before let $y = f_1(x)$ be the equation to the demand curve; $y = f_2(x)$ that to the supply curve; and that to the monopoly revenue curve is $y = f_3(x)$, where $f_3(x) = f_1(x) - f_2(x)$ the equation to the consumers' surplus curve $y = f_4(x)$; where

$$f_4(x) = \frac{1}{x}\int_0^x f_1(a)da - f_1(x).$$

That to the total benefit curve is $y = f_5(x)$; where

$$f_5(x) = f_3(x) + f_4(x) = \frac{1}{x}\int_0^x f_1(a)da - f_2(x);$$

a result which may of course be obtained directly. That to the compromise benefit curve is $y = f_6(x)$; where $f_6(x) = f_3(x) + nf_4(x)$; consumers' surplus being reckoned in by the monopolist at n times its actual value.

To find *OL* (fig. 37), that is, the amount the sale of which will afford the maximum monopoly revenue, we have the equation

$$\frac{d}{dx}\{xf_3(x)\} = 0; \text{ i.e. } f_1(x) - f_2(x) = x\{f_2'(x) - f_1'(x)\};$$

the left-hand side of this equation is necessarily positive, and therefore so is the right-hand side, which shows, what is otherwise obvious, that if Lq_3 be produced to cut the supply and demand curves in q_2 and q_1 respectively, the supply curve at q_2 (if included negatively) must make a greater angle with the vertical than is made by the demand curve at q_1.

To find *OW*, that is, the amount the sale of which will afford the maximum total benefit, we have

$$\frac{d}{dx}\{xf_5(x)\} = 0; \text{ i.e. } f_1(x) - f_2(x) - xf_2'(x) = 0.$$

To find *OY*, that is, the amount the sale of which will afford the maximum compromise benefit, we have

$$\frac{d}{dx}\{xf_6(x)\} = 0;$$

$$\text{i.e.} \quad \frac{d}{dx}\left\{(1-n)xf_1(x) - xf_2(x) + n\int_0^x f_1(a)da\right\} = 0;$$

$$\text{i.e.} \quad (1-n)xf_1'(x) + f_1(x) - f_2(x) - xf_2'(x) = 0.$$

If $OL = c$, the condition that OY should be greater than ON is that $\frac{d}{dx}\{xf_6(x)\}$ be positive when c is written for x in it; i.e. since $\frac{d}{dx}\{xf_3(x)\} = 0$ when $x = c$, that $\frac{d}{dx}\{xf_4(x)\}$ be positive when $x = c$; i.e. that $f_1'(c)$ be negative. But this condition is satisfied whatever be the value of c. This proves the first of the two results given at the end of V. XIV. 7; and the proof of the second is similar. (The working of these results and of their proofs tacitly assumes that there is only one point of maximum monopoly revenue.)

One more result may be added to those in the text. Let us write $OH = a$, then the condition that OY should be greater than OH is that $\frac{d}{dx}\{nf_6(x)\}$ be positive when a is written for x: that is, since $f_1(a) = f_2(a)$, that $(1-n)f_1'(a) - f_2'(a)$ be positive. Now $f_1'(a)$ is always negative, and therefore the condition becomes that $f_2'(x)$ be negative, i.e. that the supply obey the law of increasing return and that $\tan\phi$ be numerically greater than $(1-n)\tan\theta$, where θ and ϕ are the angles which tangents at A to the demand and supply curves respectively make with Ox. When $n = 1$, the sole condition is that $\tan\phi$ be negative: that is, OW is greater than OH provided the supply curve at A be inclined negatively. In other words, if the monopolist regards the interest of consumers as identical with his own, he will carry his production further than the point at which the supply price (in the special sense in which we are here using the term) is equal to the demand price, provided the supply in the neighbourhood of that point obeys the law of increasing return: but he will carry it less far if the supply obeys the law of diminishing return.

NOTE XXIV. (p. 565). Let Δx be the probable amount of his production of wealth in time Δt, and Δy the probable amount of his consumption. Then the discounted value of his future services is $\int_0^T R^{-t}\left(\frac{dx}{dt} - \frac{dy}{dt}\right) dt$; where T is the maximum possible duration of his life. On the like plan the past cost of his rearing and training is $\int_{-T'}^{0} R^{-t}\left(\frac{dy}{dt} - \frac{dx}{dt}\right) dt$, where T' is the date of his birth. If we were to assume that he would neither add to nor take from the material wellbeing of a country in which he stayed all his life, we should have $\int_{-T}^{T} R^{-t}\left(\frac{dx}{dt} - \frac{dy}{dt}\right) dt = 0$; or, taking the starting-point of time at his birth, and $l = T' + T =$ the maximum possible length of his life, this assumes the simpler form, $\int_0^l R^{-t}\left(\frac{dx}{dt} - \frac{dy}{dt}\right) dt = 0$.

To say that Δx is the probable amount of his production in time Δt, is to put shortly what may be more accurately expressed thus:—let $p_1, p_2, \ldots$ be the chances that in time Δt he will produce elements of wealth $\Delta_1 x, \Delta_2 x, \ldots$, where $p_1 + p_2 + \cdots = 1$; and one or more of the series $\Delta_1 x, \Delta x, \ldots$ may be zero; then

$$\Delta x = p_1 \Delta_1 x + p_2 \Delta_2 x + \cdots.$$

The End